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CHART FOR CRITICAL COMPRESSIVE STRESS OF FLAT RECTANGULAR PLATES

By H. N. Hill
Aluminum Company of America

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SUMMARY

A chart is presented for the coefficient K in the formula for the critical compressive stress for flat rectangular plates uniformly compressed in one direction. The chart applies to various combinations of fixed, simply supported, and free edges.

Chart for K in Formula for Critical Stress

The theoretical stress at which a rectangular flat plate will buckle elastically, when uniformly compressed in one direction, can be expressed (see reference 1, p. 331)

$$\sigma_{cr} = K \frac{E}{(1-\mu^2)} \left(\frac{t}{b}\right)^2$$

where σ_{cr} critical stress (at which buckling occurs), pounds per square inch

E modulus of elasticity of the material of the plate, pounds per square inch

μ Poisson's ratio of the material of the plate

t thickness of the plate, inch

b width of the plate (normal to the direction of compression), inch

K a coefficient depending on the ratio of length to width of the plate (L/b), the conditions of restraint at the edges of the plate, and in some cases on the Poisson's ratio of the material of the plate

In designing structures involving flat plates, a chart from which K values for the above equation could be obtained, for various conditions of edge restraint, should prove a convenience. It would be impossible to include all the possible variations or combinations of edge conditions, but by defining three definite edge conditions, and considering the various combinations of these conditions that might occur, certain limiting cases are obtained. Other cases involving intermediate conditions of edge restraint will be bounded by several of these limiting cases. The definite edge conditions considered are:

1. Free edge - an edge about which the plate is free to rotate and at which the plate is free to deflect.

2. Supported edge - an edge about which the plate is free to rotate but at which there can be no deflection.

3. Fixed edge - an edge at which there can be no deflection and about which there can be no rotation of the plate.

The buckling stress for a rectangular flat plate subjected to uniform compression in one direction, with the unloaded edges free, can be determined from the column formulas. For plates, "E" in the column formulas should be replaced by the "plate modulus" $[E/(1-\mu^2)]$ (reference 2, p. 475).

For plates with other edge conditions K values to be used in the equation for critical stress can be obtained from the curves plotted in figure 1. Curves are given for five combinations of conditions of restraint at the unloaded edges. For each of these combinations two conditions of the loaded edges have been considered. Theoretically, curves for all cases but cases 1 and 1a should be composed of a series of scallops, each scallop representing buckling into a certain number of half waves. For practical reasons, however, these scallops have been eliminated. The curves shown in figure 1 are the envelope curves drawn tangent to the scallops.

The curves of figure 1 represent various degrees of approximation to the theoretically correct values, although in every case the curves are sufficiently accurate for design purposes. The sources from which K values for the different cases were obtained are given in table I, together with pertinent remarks concerning the plotting of the curves. For those cases for which K values were obtained directly by the writer, the major elements of the method of solution employed are given in table I and the solutions are included in an appendix to this report.

For the cases in which one unloaded edge of the plate is free (1, 1a, 2, and 2a), the K values are based on a Poisson's ratio value of 1/3, which is the accepted value for aluminum alloys. In some instances, a difference between Poisson's ratio values of 1/4 and 1/3 is responsible for a difference of as much as 15 percent in the K value. For cases other than those involving one free edge, the curves of figure 1 are applicable to any elastic material, the K values being independent of Poisson's ratio.

The curves of figure 1 demonstrate the importance of the condition of restraint of the loaded edges in determining the buckling stress of rectangular flat plates having L/b ratios less than 3.

Aluminum Company of America,
Aluminum Research Laboratories,
New Kensington, Pa., April 16, 1940.

TABLE I

SOURCE OF K VALUES PLOTTED IN FIGURE 1

Case	Source and Remarks
1	Solution from Timoshenko's "Theory of Elastic Stability," pp. 339-340, and the values calculated by the writer for a Poisson's ratio value of 1/3.
1a	Approximate solution by the writer, using the energy method and the deflection function $w = A(1-\cos 2\pi x/L)(y + By^7)$, in which $B = f(b,\mu)$ is determined from boundary conditions.
2	Solution from Timoshenko's "Theory of Elastic Stability," p. 341, and values calculated by the writer for a Poisson's ratio value of 1/3.
2a	Approximate solution by the writer, using the energy method and deflection functions $w = A(1-\cos 2\pi x/L)(1-\cos ny/b)$ for buckling in one half wave, and $w = A(\sin 2Jx/L + Bx^3/L^3)(1-\cos ny/B)$ for buckling in any even number of half waves. Coefcients n, B, and J are determined from boundary conditions.
3	Solution from Timoshenko's "Theory of Elastic Stability," pp. 329-332.
3a	Solution from Timoshenko's "Theory of Elastic Stability," pp. 363-364.
4	Solution by the writer following the method employed by Timoshenko in "Theory of Elastic Stability," pp. 337-342.
4a	The relation of this curve to that for case 4 is estimated from the relations between the curves for cases 3 and 3a and cases 5 and 5a.
5	Solution from Timoshenko's "Theory of Elastic Stability," pp. 344-345.
5a	Solution from "Buckling of Compressed Rectangular Plates with Built-in Edges," by J. L. Maulbetsch, Journal of Applied Mechanics, June 1937.

APPENDIX

Consider a flat plate of length L and width b, subjected to a uniform end compression (in the L direction) of N_x per unit width. Consider the axes disposed as indicated in figure 2. If the deflection of the plate out of its original plane is denoted by w, the differential equation for the buckled plate may be expressed (reference 1, p. 337)

$$\frac{\partial^4 w}{\partial x^4} + 2\frac{\partial^4 w}{\partial x^2 \partial y^2} + \frac{\partial^4 w}{\partial y^4} = -\frac{N_x}{D}\frac{\partial^2 w}{\partial x^2} \qquad (1)$$

where $D = \dfrac{Et^3}{12(1-\mu^2)}$ rigidity of the plate per unit width

The various boundary conditions for the edges of the plate parallel to the x axis may be expressed as follows (reference 1, pp. 298-300):

For "fixed" edges,

$$w = 0 \quad \text{and} \quad \frac{\partial w}{\partial y} = 0$$

For "supported" edges,

$$w = 0 \quad \text{and} \quad \frac{\partial^2 w}{\partial y^2} + \mu\frac{\partial^2 w}{\partial x^2} = 0$$

For "free" edges,

$$\frac{\partial^2 w}{\partial y^2} + \mu\frac{\partial^2 w}{\partial x^2} = 0 \quad \text{and} \quad \frac{\partial^3 w}{\partial y^3} + (2-\mu)\frac{\partial^3 w}{\partial y \partial x^2} = 0$$

In cases in which load is applied to "supported" edges, the problem of the buckling of a flat plate under edge compression can be solved by integration of the differential equation, obtaining the constants of integration from the boundary conditions. Such a method has been employed by Timoshenko for obtaining the solution for cases 1, 2, and 5, as well as for certain cases involving intermediate conditions of edge restraint (reference 1, pp. 337-350). In obtaining a solution for case 4, the writer has followed the integration method of Timoshenko, determining integration constants consistent with the boundary conditions for this case.

For cases in which the loaded edges are "fixed," the above-mentioned method of integrating the differential equation cannot be used, and it is convenient to resort to one of the approximate methods for obtaining a solution. The writer has obtained approximate solutions for cases 1a and 2a by application of the "energy" method. This method also has been used extensively by Timoshenko (reference 1, p. 81). The "energy equation," obtained by equating expressions for the work of external forces and the energy of bending may be written:

$$N_x = D \frac{\int_0^b \int_0^L \left\{ \left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} \right)^2 - 2(1-\mu)\left[\frac{\partial^2 w}{\partial x^2} \frac{\partial^2 w}{\partial y^2} - \left(\frac{\partial^2 w}{\partial x \, \partial y} \right)^2 \right] \right\} dx\, dy}{\int_0^b \int_0^L \left(\frac{\partial w}{\partial x} \right)^2 dx\, dy} \qquad (2)$$

The critical value for N_x is obtained by solving this equation, using an expression for the deflection w, which is some function of x and y and satisfies the boundary conditions of the case being considered. Except in the very improbable instance when the assumed expression for the deflection exactly defines the configuration of the buckled plate, the critical load values obtained by the energy method are always higher than the true values (reference 1, p. 81). The degree of approximation of the energy method can sometimes be improved by including a parameter in the expression for the deflected surface, and evaluating this parameter so as to minimize the value determined for the critical load.

Case 1a

Loaded edges ($x = 0$ and $x = L$) fixed; one unloaded edge ($y = 0$) supported and the other unloaded edge ($y = b$) free.

The boundary conditions for this case are:

[1] $w = 0$ when $x = 0$ and when $x = L$

[2] $\frac{\partial w}{\partial x} = 0$ when $x = 0$ and when $x = L$

[3] $w = 0$ when $y = 0$

$$[4] \quad \frac{\partial^2 w}{\partial y^2} + \mu \frac{\partial^2 w}{\partial x^2} = 0 \qquad \text{when} \quad y = 0$$

$$[5] \quad \frac{\partial^2 w}{\partial y^2} + \mu \frac{\partial^2 w}{\partial y^2} = 0 \qquad \text{when} \quad y = b$$

$$[6] \quad \frac{\partial^3 w}{\partial y^3} + (2-\mu) \frac{\partial^3 w}{\partial y \partial x^2} \qquad \text{when} \quad y = b$$

To solve this problem by the energy method, assume an expression for the buckled surface of the form,

$$w = f(x)\ F(y)$$

From boundary condition [5]

$$F(x)\ F''(b) = -\ \mu f''(x)\ F(b)$$

(The primes indicate the order of the partial derivatives.)

This relation may be written

$$\frac{f(x)}{f''(x)} = -\mu \frac{F(b)}{F''(b)}$$

Similarly, from boundary condition 6

$$f(x)\ F'''(b) = -(2-\mu)\ f''(x)\ F'(b)$$

or

$$\frac{f(x)}{f''(x)} = -(2-\mu) \frac{F'(b)}{F'''(b)}$$

Combining these two equations

$$\mu F(b)\ F'''(b) = (2-\mu)\ F'(b)\ F''(b) \tag{3}$$

Boundary conditions 3 and 4 are satisfied by an expression of the form

$$w = f(x)\ (y+By^n)$$

$$\text{i.e.} \quad F(y) = (y+By^n) \tag{4}$$

where $n \neq 1$

Substituting equation (4) and its derivatives into equation (3) and solving for B

$$B = \frac{H}{b^{n-1}} \tag{5}$$

where

$$H = \frac{2-\mu n+\mu}{2(\mu n-\mu-n)} \tag{6}$$

The boundary conditions [1] and [2], defining the conditions at the loaded edges, are satisfied by

$$f(x) = A\left(1-\cos\frac{2\pi x}{L}\right)$$

The expression for the shape of the buckled plate is then

$$w = A\left(1-\cos\frac{2\pi x}{L}\right)(y+By^n) \tag{7}$$

Substituting equation (7) and its partial derivatives into the energy equation (2) and reducing, yields for the critical value of N_x

$$N_{xcr} = C\,\frac{\pi^2 D}{b^2} \tag{8}$$

where

$$C = \frac{4}{(L/b)^2} + \frac{\left(\frac{L}{b}\right)^2 \frac{1}{\pi^2}\left[\frac{3}{4}\,\frac{n^2(n-1)^2}{(2n-3)}\,H^2\right]}{\pi^2\left[\frac{1}{3}+\frac{2}{n+2}\,H+\frac{1}{2n+1}\,H^2\right]}$$
$$+\frac{2\left\{(1-\mu)+[(1+\mu)(n+1)-(n-1)]H+\left[(1-\mu)n-\frac{n(n-1)}{(2n-1)}\right]H^2\right\}}{\pi^2\left[\frac{1}{3}+\frac{2}{n+2}\,H+\frac{1}{2n+1}\,H^2\right]} \tag{9}$$

Substituting for μ in equations (6) and (9), the accepted value for aluminum alloys $\mu = 1/3$

$$C = \frac{4}{(L/b)^2} + \left(\frac{L}{b}\right)^2 \frac{3}{4\pi^4} \frac{\left[\frac{n^4 - 2n^3 + n^2}{2n - 3}\right] H^2}{\left[\frac{1}{3} + \frac{2}{n+2} H + \frac{1}{2n+1} H^2\right]}$$

$$+ \frac{2}{3\pi^2} \frac{\left[2 + (5-n) H + \frac{n^2+n}{2n-1} H^2\right]}{\left[\frac{1}{3} + \frac{2}{n+2} H + \frac{1}{2n+1} H^2\right]} \tag{9a}$$

and

$$H = -\frac{7-n}{2+4n} \tag{6a}$$

The expression chosen for the shape of the buckled plate (equation (7)) is based on the assumption that the plate will buckle into one half wave regardless of the L/b ratio. It has been demonstrated that plates loaded through supported edges, and with one unloaded edge supported and the other free (case 1), will buckle in this manner (reference 1, p. 339). There is no reason to believe that fixing the loaded edges would change this characteristic of the buckling of such a plate. That buckling will always occur in the form of one half wave means that the minimum value of C will occur at an infinite value of L/b. Referring to equation (9a), it can be seen that a minimum value for C will occur at some finite value of L/b unless the coefficient of the second term is zero. The only significant value of the parameter n which will reduce this coefficient to zero is $n = 7$. Equation (9a) then reduces to

$$C = \frac{4}{(L/b)^2} + 0.406 \tag{10}$$

Timoshenko gives an approximate expression for C for case 1 (reference 1, p. 340), which involves two terms, corresponding to those in equation (1). While this approximate expression is advocated for long plates, it also gives values of sufficient accuracy for short plates. The only significant difference between equation (10) and the corresponding equation for case 1 lies in the term involving $(L/b)^2$. For case 1a this term is four times as large as for case 1.

Equation (8) expressed in terms of critical stress becomes

$$\sigma_{cr} = C\,\frac{\pi^2}{12}\,\frac{E}{1-\mu^2}\left(\frac{t}{b}\right)^2$$

Therefore, the coefficient K, for which values are plotted in figure 1, may be expressed

$$K = \frac{\pi^2}{12}\,C$$

Case 2a

Loaded edges $(x = 0$ and $x = L)$ fixed; one unloaded edge $(y = 0)$ fixed and the other unloaded edge $(y = b)$ free.

The boundary conditions for this case are:

[1]	$w = 0$	when $x = 0$ and when $x = L$
[2]	$\dfrac{\partial w}{\partial x} = 0$	when $x = 0$ and when $x = L$
[3]	$w = 0$	when $y = 0$
[4]	$\dfrac{\partial w}{\partial y} = 0$	when $y = 0$
[5]	$\dfrac{\partial^2 w}{\partial y^2} + \mu\dfrac{\partial^2 w}{\partial x^2} = 0$	when $y = b$
[6]	$\dfrac{\partial^3 w}{\partial y^3} + (2-\mu)\,\dfrac{\partial^3 w}{\partial y\,\partial x^2} = 0$	when $y = b$

Assume an expression for the buckled surface of the form

$$w = f(x)\ F(y)$$

Since boundary conditions [5] and [6] are the same as for case 1a (a free edge), equation (3) is also applicable to this case. Boundary conditions [3] and [4] are satisfied by an expression of the form

$$w = f(x)\left(1-\cos\frac{ny}{b}\right)$$

i.e.,

$$F(y) = \left(1-\cos\frac{ny}{b}\right) \tag{11}$$

Substituting equation (11) and its derivatives into equation (3) and solving for n

$$\cos n = -\frac{\mu}{2(1-\mu)} \tag{12}$$

for $\mu = 1/3$ (for aluminum alloys)

$$n = 1.8343$$

While in cases 1 and 1a the plate can buckle into only one half wave, in cases 2 and 2a the buckled configuration may contain one or more half waves, depending on the proportions of the plate. In solving this problem, two general forms of configuration have been considered. First, it was assumed that the plate buckled into one half wave. Then a solution was obtained which covers buckling into any even number of half waves, the solution being carried out specifically for two half waves. Buckling into a number of half waves greater than 2 was not considered, since in such cases the effect of the condition of restraint at the loaded edges is negligible.

Considering the plate to buckle into one half wave, the boundary conditions [1] and [2], defining the conditions at the loaded edges, will be satisfied by the equation

$$f(x) = A\left(1-\cos\frac{2\pi x}{L}\right)$$

and the expression for the buckled surface will be

$$w = A\left(1-\cos\frac{2\pi x}{L}\right)\left(1-\cos\frac{ny}{b}\right) \tag{13}$$

Substituting equation (13) and its partial derivatives into the energy equation (2), and solving for the critical value for N_x gives (for aluminum alloys)

$$N_{x_{cr}} = C\,\frac{\pi^2 D}{b^2}$$

where

$$C = \frac{4}{\left(\frac{L}{b}\right)^2} + 0.0993\left(\frac{L}{b}\right)^2 + 0.624 \tag{14}$$

As in case 1a

$$K = \frac{\pi^2}{12} C$$

For the type of buckling involving an even number of half waves, the configuration may be approximated by the equation

$$w = A \left(\sin \frac{2Jx}{L} + \frac{Bx^3}{L^3}\right)\left(1-\cos \frac{ny}{b}\right) \qquad (15)$$

where J and B are coefficients to be evaluated from the boundary conditions. In this instance the axes are taken as shown in figure 3, and the integrals with respect to dx in the energy equation (2) are taken between the limits 0 and $L/2$. Boundary conditions [1] and [2] become

$$(1) \quad w = 0 \quad \text{when} \quad x = \pm L/2$$

$$(2) \quad \frac{\partial w}{\partial x} = 0 \quad \text{when} \quad x = \pm L/2$$

From boundary condition 1

$$B = -8 \sin J$$

Evaluating J from boundary condition [2], yields the transcendental equation

$$J = 3 \tan J \qquad (16)$$

The various roots of this equation give values corresponding to various even numbers of half waves of the buckled configuration. The lowest root corresponds to a buckled form of two half waves. This root is

$$J = 4.0782$$

The critical value for N_x can be found as before, by substituting equation (15) and its partial derivatives into the energy equation. Such a substitution, in which the coefficient J has been replaced by its lowest root, gives for the critical load for aluminum alloys $(\mu = 1/3)$.

$$N_{x_{cr}} = C \frac{\pi^2 D}{b^2}$$

where

$$C = \frac{5.626}{\left(\frac{L}{b}\right)^2} + 0.0178\left(\frac{L}{b}\right)^2 + 0.638 \qquad (17)$$

As before, the K values shown in the curves are related to the C values as

$$K = \frac{\pi^2}{12} C$$

Case 4

Loaded edges $(x = 0$ and $x = L)$ supported; one unloaded edge $(y = 0)$ fixed and the other unloaded edge $(y = b)$ supported.

The boundary conditions for this case are:

[1] $w = 0$ when $x = 0$ and when $x = L$

[2] $\frac{\partial^2 w}{\partial x^2} + \mu \frac{\partial^2 w}{\partial y^2} = 0$ when $x = 0$ and when $x = L$

[3] $w = 0$ when $y = 0$

[4] $\frac{\partial w}{\partial y} = 0$ when $y = 0$

[5] $w = 0$ when $y = b$

[6] $\frac{\partial^2 w}{\partial y^2} + \mu \frac{\partial^2 w}{\partial x^2} = 0$ when $y = b$

Following the method of solution of the differential equation (1) described by Timoshenko (reference 1, p. 337) in which the solution is taken in the form

$$w = f(y) \sin \frac{m\pi x}{L}$$

the general expression for $f(y)$ can be written

$$f(y) = C_1 e^{-\alpha y} + C_2 e^{+\alpha y} + C_3 \cos \beta y + C_4 \sin \beta y$$

in which

$$\alpha = \sqrt{\frac{m^2\pi^2}{L^2} + \sqrt{\frac{N_x}{D} \frac{m^2\pi^2}{L^2}}} \quad \text{and} \quad \beta = \sqrt{-\frac{m^2\pi^2}{L^2} + \sqrt{\frac{N_x}{D} \frac{m^2\pi^2}{L^2}}}$$

and C_1, C_2, C_3, and C_4 are coefficients to be evaluated from the boundary conditions. From boundary conditions [3] and [4] it follows that

$$C_1 = -\frac{\alpha C_3 - \beta C_4}{2\alpha} \quad \text{and} \quad C_2 = -\frac{\alpha C_3 + \beta C_4}{2\alpha}$$

and $f(y)$ can be written

$$f(y) = A(\cos\beta y - \cosh\alpha y) + B\left(\sin\beta y - \frac{\beta}{\alpha}\sinh\alpha y\right)$$

From boundary conditions [5] and [6] two simultaneous equations are obtained. The critical stress can then be obtained by equating to zero the determinant of these equations in A and B. This manipulation results in the transcendental equation

$$(\alpha^2 + \beta^2)\sin\beta b\cosh\alpha b = \left(\alpha\beta + \frac{\beta^3}{\alpha}\right)\cos\beta b\sinh\alpha b$$

For given values of L/b, this equation can be solved for N_{xcr}, using for the number of half waves (m) the integer which results in minimum values. The critical stress can be expressed in the form

$$\sigma_{cr} = K\frac{E}{1-\mu^2}\left(\frac{t}{b}\right)^2$$

and the coefficient K thus evaluated for different ratios of L/b.

REFERENCES

1. Timoshenko, Stephen: Theory of Elastic Stability. McGraw-Hill Book Co., Inc. (New York), 1930.

2. Timoshenko, Stephen: Strength of Materials. D. Van Nostrand Co., Inc. (New York), 1930.

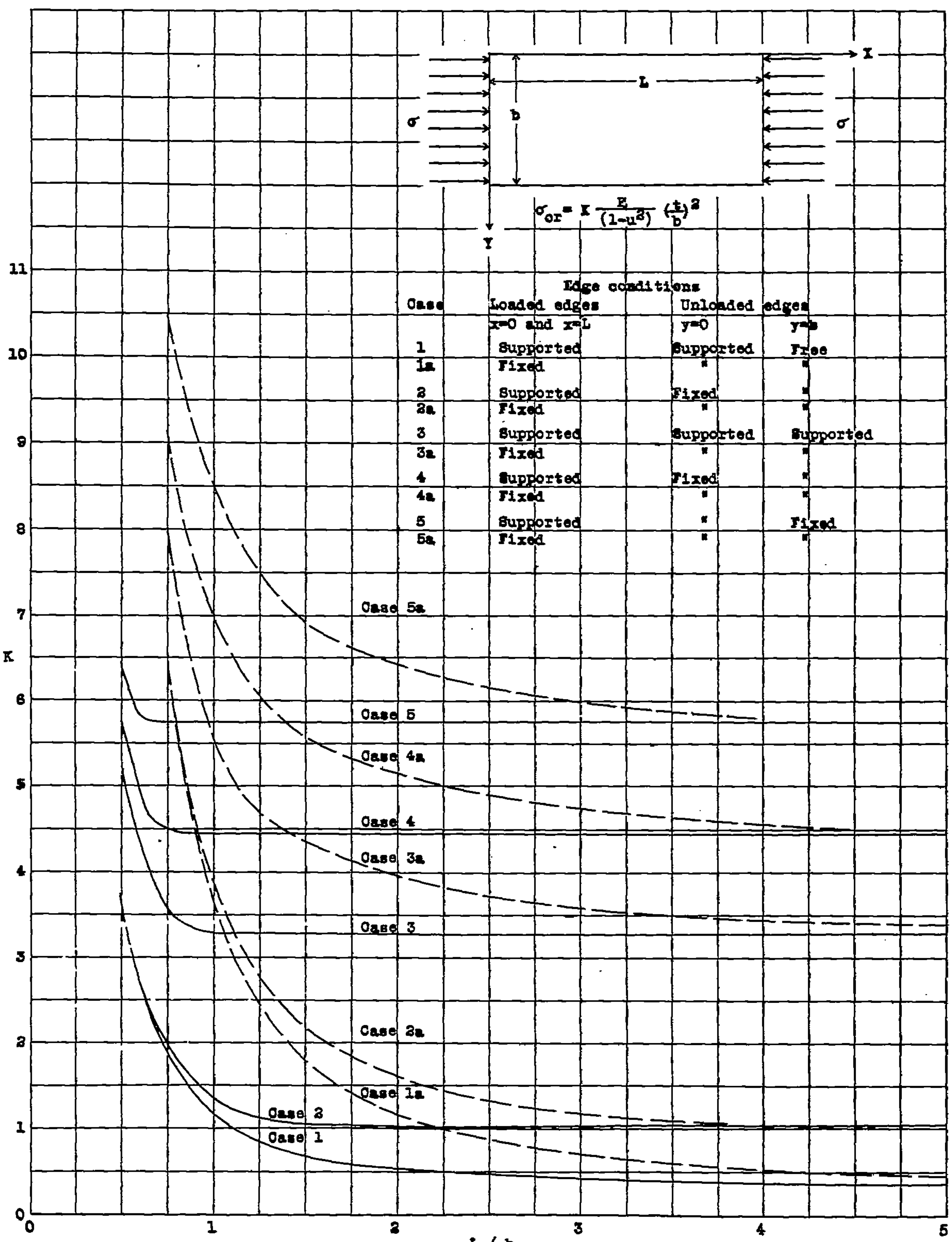

Figure 1.- Note: For L/b > 5, Case 1, $K = \frac{\pi^2}{12}(\frac{1}{(L/b)^2} + 0.408)$. Case 1a, $K = \frac{\pi^2}{12}(\frac{4}{(L/b)^2} + 0.406)$

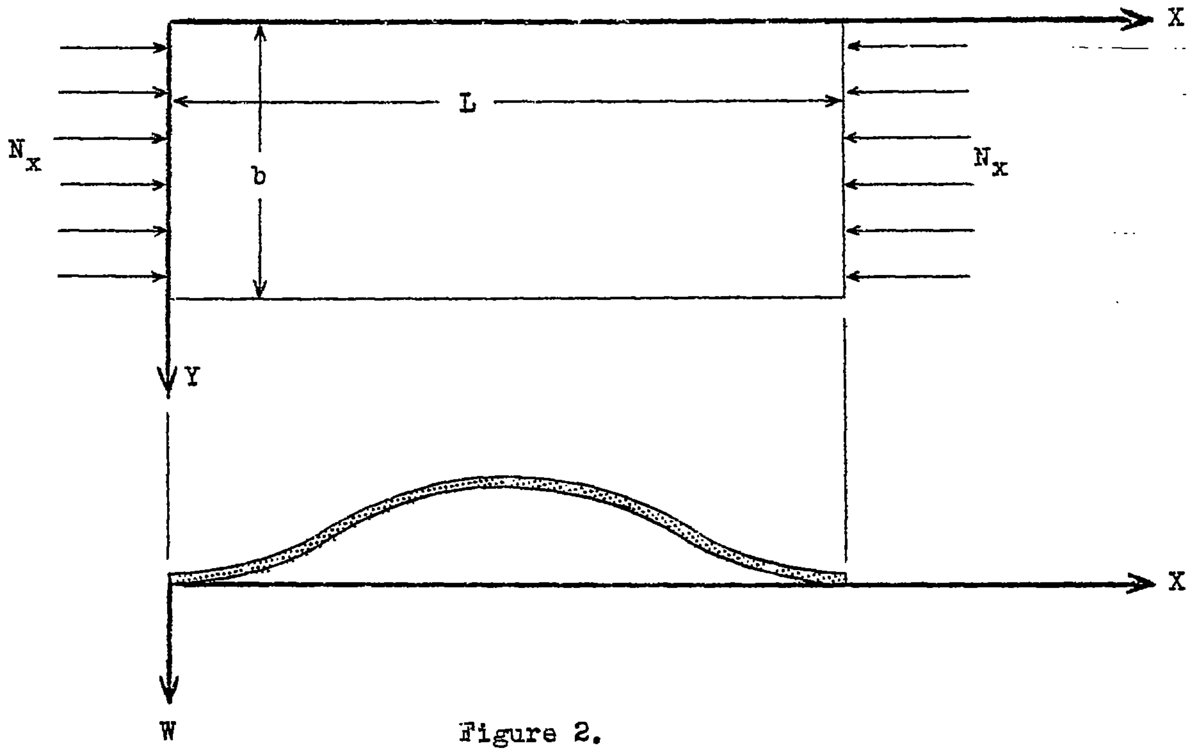

Figure 2.

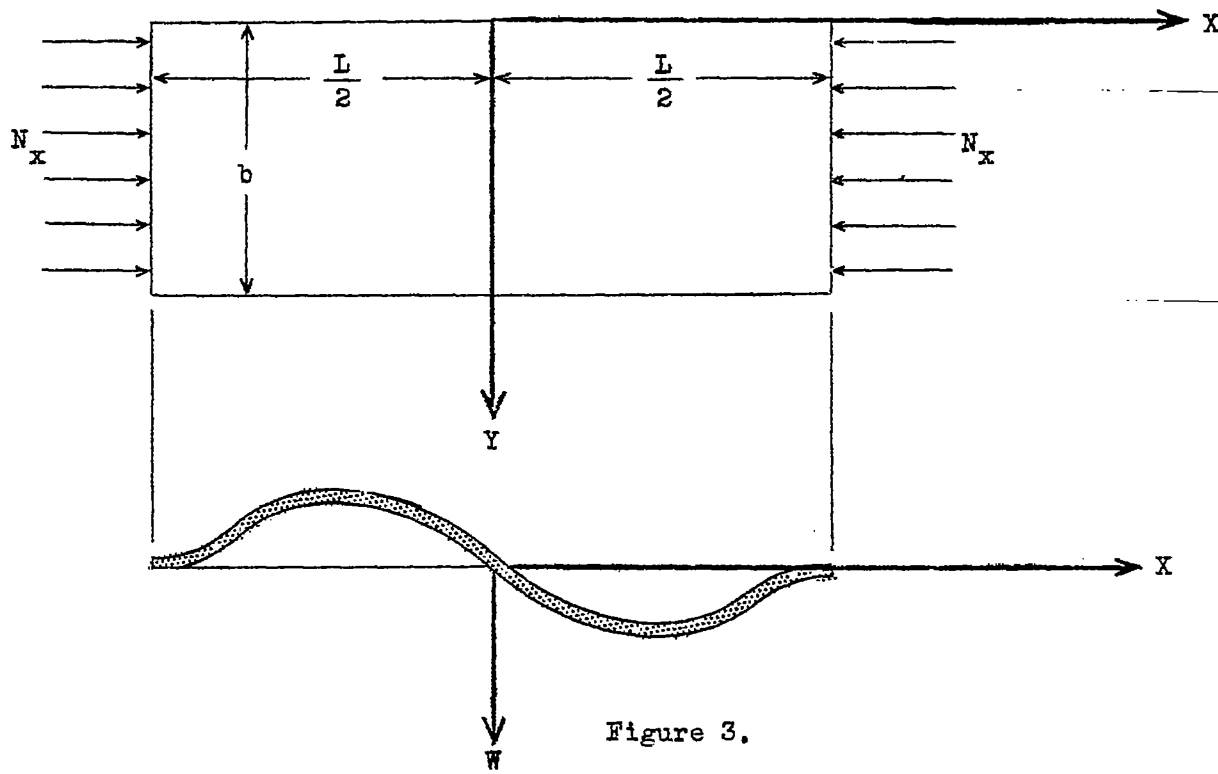

Figure 3.